CS559 – Lecture 11
Polygons, Transformations

These are course notes (not used as slides)
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With some slides adapted from the notes of Stephen Chenney

Final version (after class)

Today

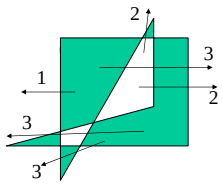
- Polygon rules (since I got them wrong last time)
- Homogeneous coordinates
- Working with transformations / composition
- Hierarchies / Matrix Stacks
- Transformations in 3D
  - Coordinate systems in 3D
  - Rotations
  - Projections (3D->2D)

General Polygons?

- Inside / Outside not obvious for general polygons
- Usually require simple polygons
  - Convex (easy to break into triangles)
- For general case, three common rules:
  - Non-exterior rule: A point is inside if every ray to infinity intersects the polygon
  - Non-zero winding number rule: trace around the polygon, count the number of times the point is circled (+1 for clockwise, -1 for counter clockwise). Odd winding counts = inside (note: I got this wrong in class)
  - Parity rule: Draw a ray to infinity and count the number or edges that cross it. If even, the point is outside, if odd, it's inside (ray can't go through a vertex)

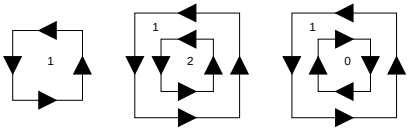
Parity

- Any point, take any ray (that doesn't go through a vertex)
- Odd number of crossings = inside
- Even number of crossings = outside



Power Point uses this rule!

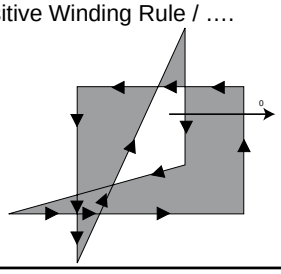
Winding Numbers



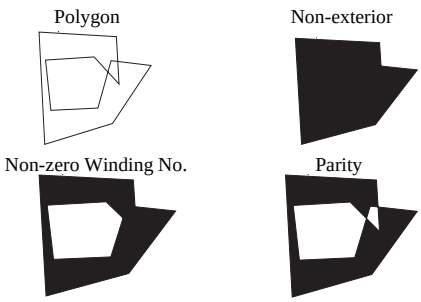
- Count the number of times a point is circled counter clockwise
  - Clockwise counts negative
- Can pick any ray from point and count left/right
  - Right (relative to away direction) = CCW = +1
  - Left = CW = -1

Non-Zero Winding Rule

- Any non-zero winding is "inside"
- What Adobe Illustrator does
- Odd Winding Rule / Positive Winding Rule / ….



Inside/Outside Rules

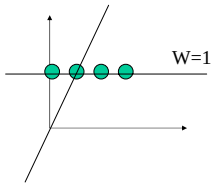


Homogeneous Coordinates

- Big idea for graphics – really important
  - Will be used for several things – translation is just 1
- Basic idea: add an extra coordinate
  - 2D becomes 3D (3x3 matrices)
  - 3D becomes 4D (4x4 matrices)
- Convert "back" from homogeneous coordinates by division
  - (x,y) -> (x,y,1)
  - (x,y,w) -> (x/w, y/w)
- Projection
  - Many points in higher dim space = 1 point in lower dim space
- For now, just make w=1

Homogeneous Coordinates

- "Normal" space is a subspace
  - W = 1
- Think about 1D case (so embed into 2D x,w)
- Many equivalent points (projection)

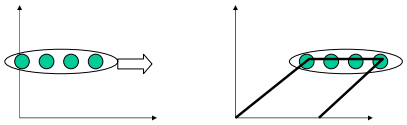


Only 1D Linear operation is scale

(about origin)

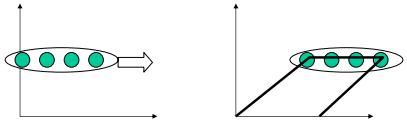
Translation in Homogeneous Coords

- 1D Translation = 2D Skew



Translation in Homogeneous Coords

- Translate in 2D = Skew in 3D
  - Deck of cards



$$trans(x,y) = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

What about other linear ops

- Just add an extra coordinate
- Don't change w (unless you know what you're doing)

$$scale(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$rotate(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates

- Makes translation (affine transforms) linear
- Need to work in higher dimensional space

- Useful for lots of other things
  - Viewing (perspective)

Matrices as Coordinate Systems

- Where does X axis go?
- Where does Y axis go?
- Where does origin go?

- Assumes that bottom row is [0 0 1]

- Can you scale by changing w?
  - Yes, but often we prefer to renormalize so bottom right number is 1

Composing Transformations

- Order matters!
  - Scale / rotate vs. rotate/scale

- Can implement by multiplying matrices
  - $T_1\ T_2\ T_3\ \mathbf{x} = (T_1\ T_2\ T_3)\ \mathbf{x}$

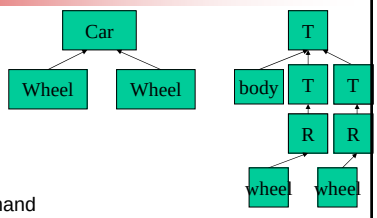
Why Compose?

- Rotate about a point
  - $T_c\ R\ T_{-c}\ x$
- Scale along an axis
  - Move point to origin
  - Align axis w/major axis
  - Scale
  - Put things back
  - $T_c\ R_\theta\ S\ R_{-\theta}\ T_{-c}\ x$

Hierarchical coordinate Systems

- Car
  - Wheel
  - Wheel

- Person
  - Head / Neck
  - Arm / forearm / hand



Matrix Stack

- Multiply things onto the top
- Top is “current” coordinate system
- Push (copy the top) if you’ll come back
- Pop to go back

- Think about it as moving the coordinate system
- Top of stack is “current coordinate system”
  - Where we will draw
- Transformations change current coord system
  - Or change the objects that we are going to draw

Matrix Stack Example

- Draw Car = …. Push trans wheel pop …
- Push trans – draw car – pop push trans – draw car

3D

- 3D coordinate system & handedness
- Prefer right-handed coordinate systems
- Right-hand rule

What happens in 3D?

- 4D Homogeneous Points
  - 4x4 matrices
- Basic transforms are the same
  - Translate
  - Scale
  - Skew
- Rotation is different
  - Rotation in 3D is more complicated?

What is a rotation?

- A transformation that:
  - Preserves distances between points
  - Preserves the zero
  - Preserves "handedness" (in 2D clockwiseness)
- A subset of linear transformations
- Some things that come out of these:
  - Axes remain perpindicular
  - Axes remain unit length
  - Cross product holds

Parameterizing Rotations

- Rotations are Linear Transformations
  - 2x2 matrix in 2D
  - 3x3 matrix in 3D
- The set of rotations = set of OrthoNormal Matrices
- Inconvenient way to deal with them
  - Can't work with them directly
  - Not stable (small change makes it not a rotation)
- Is there an easier way to parameterize the set?

Measuring rotation in 2D

- Pick 1 point (1,0)
- Any rotation must put this on a circle
- If you know where this point goes, can figure out any other point
  - Distances (w/point & origin) + handedness says where things go
- Parameterize rotations by distance around circle
  - Angle
- Issues with wrap around
  - Many different angles = same rotation

Much harder in 3D

- Any point can go to a sphere
- That one point doesn't uniquely determine things

- No vector in R^n can compactly represent rotations
  - Singularities
    - nearby rotations / far away numbers
    - Nearby numbers / far away rotations
- Hairy-Ball Theorem
  - Any parameterization of 3D rotations in R^n will have singularities

Representation of 3D Rotations

- Two Theorems of Euler
  - Any rotation can be represented by a single rotation about an arbitrary axis (axis-angle form)
  - Any rotation can be represented by 3 rotations about fixed axes (Euler Angle form)
    - XYZ, XZX, any non-repeating set works
    - Each set is different (gets different singularities)

- Building rotations
  - Pick a vector (for an axis)
  - Pick another perpindicular vector (or make one w/cross product)
  - Get third vector by cross product

Euler Angles

- Pick convention
  - Are axes local or global?
  - Local: roll, pitch, yaw
  - What order?
- Apply 3 rotations
- Good news: 3 numbers
- Bad news:
  - Can't add, can't compose
  - Many representations for any rotation
  - Singularities