Technical notes

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1 Derivation of corotational elasticity

The simple material model known as linear elasticity uses the following definition for the energy density function

$$\Psi_{LE} = \mu \|\boldsymbol{\epsilon}\|_F^2 + \frac{\lambda}{2}\text{tr}^2\boldsymbol{\epsilon}$$

where μ and λ are the Lamé coefficients and $\boldsymbol{\epsilon}$ is the so-called small strain tensor, defined as

$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I}.$$

The linear elasticity model is known to produce jarring artifacts, especially under large deformation due to the fact it is not rotationally invariant. That is, the values of the energy associated with two different deformed states, each of which is a rotation of the other (i.e. $\boldsymbol{\phi}_1(\boldsymbol{X}) = \mathbf{R}_0\boldsymbol{\phi}_2$, or $\mathbf{F}_1(\boldsymbol{X}) = \mathbf{R}_0\mathbf{F}_1(\boldsymbol{X})$, for any $\boldsymbol{X}$) do *not* coincide, which contradicts the intuitive property that a rigid rotation of an object keeps the internal elastic forces unchanged, with respect to the body's frame of reference.

The corotational elasticity model improves on the physical plausibility of linear elasticity, by evaluating the strain tensor in a local coordinate frame that matches the world space rotation at any given material point. This rotation is obtained from the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{RS}$, where $\mathbf{R}$ is a rotation matrix and $\mathbf{S}$ is a symmetric tensor. We define the modified strain measure used in corotated elasticity by replacing $\mathbf{F}$ with $\mathbf{R}^T\mathbf{F}$ ($= \mathbf{S}$, the un-rotated version of the deformation gradient) in the expression for the small strain tensor

$$\hat{\boldsymbol{\epsilon}}(\mathbf{F}) = \boldsymbol{\epsilon}(\mathbf{R}^T\mathbf{F}) = \frac{1}{2}\left(\mathbf{R}^T\mathbf{F} + (\mathbf{R}^T\mathbf{F})^T\right) - \mathbf{I} = \mathbf{S} - \mathbf{I}$$

Substituting $\hat{\boldsymbol{\epsilon}}$ for $\boldsymbol{\epsilon}$ in the energy definition, we obtain the energy density function for corotational linear elasticity:

$$\Psi = \mu\|\mathbf{S} - \mathbf{I}\|_F^2 + \frac{\lambda}{2}\text{tr}^2(\mathbf{S} - \mathbf{I}) \tag{1}$$

In addition to this definition, we may also use the following two expressions which can be shown to be equivalent to equation (1) using the properties of the trace and the Frobenius norm:

$$\Psi = \mu\|\mathbf{F} - \mathbf{R}\|_F^2 + \frac{\lambda}{2}\text{tr}^2(\mathbf{R}^T\mathbf{F} - \mathbf{I}), \tag{2}$$

$$\text{or} \quad \Psi = \mu\|\mathbf{\Sigma} - \mathbf{I}\|_F^2 + \frac{\lambda}{2}\text{tr}^2(\mathbf{\Sigma} - \mathbf{I}) \tag{3}$$

In the last expression $\mathbf{\Sigma}$ is the diagonal matrix containing the singular values from the Singular Value Decomposition $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ of the deformation gradient ($\mathbf{U}$ and $\mathbf{V}$ are 3×3 rotation matrices). Equation (3) also reveals that the energy only depends on a subset of the information contained in the 9 entries of $\mathbf{F}$, namely Ψ is only a function of the 3 singular values. It also indicates that corotational linear elasticity is isotropic: The energy of a given deformation is not only invariant under world-space rotations, but also independent of material-space rotations, or formally $\Psi[\boldsymbol{\phi}(\boldsymbol{X})] = \Psi[\mathbf{R}_1\boldsymbol{\phi}(\mathbf{R}_2\boldsymbol{X})]$ for any rotations $\mathbf{R}_1, \mathbf{R}_2$. In simpler terms, the energy produced by stretching or compressing along a given direction depends only on the stretch or compression ratio, not the orientation of the axis of deformation.

We can also use the previous definitions of the energy to obtain an expression of the 1st Piola-Kirchhoff stress tensor. First, we will need the following 2 lemmas:

Lemma 1. *If $\mathbf{R}$ is an orthogonal matrix, then $\mathbf{R}^T\delta\mathbf{R}$ is skew-symmetric.*

Proof. If $\mathbf{R}$ is orthogonal, then

$$\mathbf{R}^T\mathbf{R} = \mathbf{I} \Rightarrow \delta(\mathbf{R}^T\mathbf{R}) = 0 \Rightarrow \delta\mathbf{R}^T\mathbf{R} + \mathbf{R}^T\delta\mathbf{R} = 0 \Rightarrow$$

$$\Rightarrow (\mathbf{R}^T\delta\mathbf{R})^T + \mathbf{R}^T\delta\mathbf{R} = 0$$

□

Lemma 2. *tr*$(\delta\mathbf{S}) =$ *tr*$(\mathbf{R}^T\delta\mathbf{F})$

Proof. We have

$$\text{tr}(\delta\mathbf{S}) = \text{tr}(\delta(\mathbf{R}^T\mathbf{F})) = \text{tr}(\delta\mathbf{R}^T\mathbf{F} + \mathbf{R}^T\delta\mathbf{F}) =$$

$$= \text{tr}(\delta\mathbf{R}^T\mathbf{R}\mathbf{S}) + \text{tr}(\mathbf{R}^T\delta\mathbf{F}) = (\mathbf{R}^T\delta\mathbf{R}) : \mathbf{S} + \text{tr}(\mathbf{R}^T\delta\mathbf{F})$$

and, from Lemma 1, we have $(\mathbf{R}^T\delta\mathbf{R}) : \mathbf{S} = 0$, since this is a contraction of a skew-symmetric matrix with a symmetric one. □

We now compute the energy differential $\delta\Psi$ form equation (2) as

$$
\begin{aligned}
\Psi &= \mu\,\text{tr}[(\mathbf{F}-\mathbf{R})^T(\mathbf{F}-\mathbf{R})] + \frac{\lambda}{2}\text{tr}^2(\mathbf{R}^T\mathbf{F}-\mathbf{I}) \\
&= \mu[\text{tr}(\mathbf{F}^T\mathbf{F}) - \text{tr}(\mathbf{R}^T\mathbf{F}) - \text{tr}(\mathbf{F}^T\mathbf{R}) + \text{tr}(\mathbf{R}^T\mathbf{R})] + \frac{\lambda}{2}\text{tr}^2(\mathbf{S}-\mathbf{I}) \\
&= \mu[\text{tr}(\mathbf{F}^T\mathbf{F}) - 2\text{tr}(\mathbf{S}) + \text{tr}(\mathbf{I})] + \frac{\lambda}{2}\text{tr}^2(\mathbf{S}-\mathbf{I}) \\
\delta\Psi &= \mu[2\text{tr}(\mathbf{F}^T\delta\mathbf{F}) - 2\text{tr}(\delta\mathbf{S})] + \lambda\,\text{tr}(\mathbf{S}-\mathbf{I})\text{tr}(\delta\mathbf{S}) \\
&\overset{Lm.2}{=} 2\mu[\text{tr}(\mathbf{F}^T\delta\mathbf{F}) - \text{tr}(\mathbf{R}^T\delta\mathbf{F})] + \lambda\,\text{tr}(\mathbf{S}-\mathbf{I})\text{tr}(\mathbf{R}^T\delta\mathbf{F}) \\
&= [2\mu(\mathbf{F}-\mathbf{R}) + \lambda\,\text{tr}(\mathbf{S}-\mathbf{I})\mathbf{R}] : \delta\mathbf{F}
\end{aligned}
$$

Since $\mathbf{P} = \partial\Psi/\partial\mathbf{F}$, we have $\delta\Psi = \mathbf{P} : \delta\mathbf{F}$, thus:

$$
\mathbf{P} = 2\mu(\mathbf{F}-\mathbf{R}) + \lambda\,\text{tr}(\mathbf{S}-\mathbf{I})\mathbf{R}
$$

$$
(\text{or}\quad \mathbf{P} = \mathbf{R}[2\mu(\mathbf{S}-\mathbf{I}) + \lambda\,\text{tr}(\mathbf{S}-\mathbf{I})\mathbf{I}])
$$

By substituting this expression for $\mathbf{P}$ into the equilibrium condition $\mathbf{div}\mathbf{P} + \boldsymbol{g} = \mathbf{0}$, we obtain the (continuous) governing equations for the static equilibrium problem. Finally, it is noteworthy that even though the energy Ψ was a function of $\mathbf{R}$, and $\mathbf{P} = \partial\Psi/\partial\mathbf{F}$, the formula for the 1st Piola-Kirchhoff stress ultimately only requires the value of $\mathbf{R}$ and *not* its derivatives. This was a consequence of the equality $(\mathbf{R}^T\delta\mathbf{R}) : \mathbf{S} = 0$ demonstrated in Lemma 2, which spares us the computation of the rotational differential $\delta\mathbf{R}$. This strategic cancellation will resurface in our analysis of the discrete governing equations in the next sections.

2 Differential of the polar decomposition

Let $\mathbf{F} = \mathbf{RS}$ be the polar decomposition of the 3×3 tensor $\mathbf{F}$, where $\mathbf{R}$ is an orthonormal rotation matrix, and $\mathbf{S}$ is symmetric. Taking differentials we get

$$
\begin{aligned}
\delta\mathbf{F} &= \delta\mathbf{R}\cdot\mathbf{S} + \mathbf{R}\delta\mathbf{S} \\
\mathbf{R}^T\delta\mathbf{F} &= (\mathbf{R}^T\delta\mathbf{R})\mathbf{S} + \delta\mathbf{S}
\end{aligned}
\tag{4}
$$

Since $\mathbf{R}$ is a rotation matrix, we have

$$
\mathbf{R}^T\mathbf{R} = I \Rightarrow \delta(\mathbf{R}^T\mathbf{R}) = 0 \Rightarrow
$$

$$
\Rightarrow \delta\mathbf{R}^T\mathbf{R} + \mathbf{R}^T\delta\mathbf{R} = 0 \Rightarrow (\mathbf{R}^T\delta\mathbf{R})^T + \mathbf{R}^T\delta\mathbf{R} = 0
$$

Since $\mathbf{R}^T\delta\mathbf{R}$ is a skew-symmetric tensor, it can be written as a cross-product operator $\mathbf{r}_\times$, where $\boldsymbol{r} = (r_1, r_2, r_3)$ and

$$
\mathbf{r}_\times = \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix}
$$

Let us name $\mathbf{W} = \mathbf{R}^T \delta\mathbf{F}$ and rewrite equation (4) as

$$\begin{aligned}
\delta\mathbf{S} &= \mathbf{W} - \mathbf{r}_\times \mathbf{S} \\
&= \mathbf{W} - \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix} \\
&= \begin{pmatrix} * & w_{12} + r_3 s_{22} - r_2 s_{23} & w_{13} + r_3 s_{23} - r_2 s_{33} \\ w_{21} - r_3 s_{11} + r_1 s_{13} & * & w_{23} - r_3 s_{13} + r_1 s_{33} \\ w_{31} + r_2 s_{11} - r_1 s_{12} & w_{32} + r_2 s_{12} - r_1 s_{22} & * \end{pmatrix}
\end{aligned}$$

(entries marked with $*$ are not important for our proof)

The symmetry of the differential $\delta\mathbf{S}$ gives rise to the equations

$$\begin{aligned}
(\delta\mathbf{S})_{32} &= (\delta\mathbf{S})_{23} \\
w_{32} + r_2 s_{12} - r_1 s_{22} &= w_{23} - r_3 s_{13} + r_1 s_{33} \\
s_{12} r_2 + s_{13} r_3 - s_{22} r_1 - s_{33} r_1 &= w_{23} - w_{32} \\
s_{11} r_1 + s_{12} r_2 + s_{13} r_3 - (s_{11} + s_{22} + s_{33}) r_1 &= w_{23} - w_{32}
\end{aligned}$$

We can see that the last equation is the first row of the system

$$(\mathbf{S} - \mathrm{tr}(\mathbf{S})\mathbf{I})\boldsymbol{r} = \begin{pmatrix} w_{23} - w_{32} \\ w_{31} - w_{13} \\ w_{12} - w_{21} \end{pmatrix} = \boldsymbol{w}$$

and the other 2 equations come from the equality of the other 2 symmetric pairs. We can then compute $\delta\mathbf{R} = \mathbf{R}\mathbf{r}_\times$ and $\delta\mathbf{S} = \mathbf{W} - \mathbf{r}_\times \mathbf{S}$.

Finally, the cross product matrix $\mathbf{r}_\times$ can be written as $\mathbf{r}_\times = -\mathcal{E} : \boldsymbol{r}$, where $\mathcal{E}$ is the alternating tensor (or Levi-Civita symbol). Similarly, the equation that defines $\boldsymbol{w}$ is written as $\boldsymbol{w} = \mathcal{E}^T : \mathbf{W}$. Combining these expressions we obtain the following exression for $\delta\mathbf{R}$, which reveals the symmetry of the derivative $\partial\mathbf{R}/\partial\mathbf{F}$:

$$\delta\mathbf{R} = \mathbf{R}\left[\mathcal{E} : \left((\mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S})^{-1}\left(\mathcal{E}^T : (\mathbf{R}^T\delta\mathbf{F})\right)\right)\right].$$

3 Computing the diagonal part of the stiffness matrix

We focus our analysis on the contribution of each individual element Ω_e to the diagonal part of the global stiffness matrix. Additionally, we will only construct the diagonal part of the matrix $\mathbf{K}_{\text{aux}}$ (denoted simply as $\mathbf{K}$ for the remaining of this section), corresponding to the auxiliary energy term Ψ_{aux}. The Laplace component of the energy yields a constant stiffness matrix for all elements, which can be added to $\mathbf{K}_{\text{aux}}$ in the end of this process.

We shall focus on computing the diagonal entry of the elemental stiffness matrix, corresponding to the j-th component of the i-th element vertex (see figure 1). Let $\delta\boldsymbol{x}$ be the 24-dimensional vector containing the stacked nodal displacements for each node in Ω_e. Additionally, set $\delta x_p^{(q)} = \delta_{ip}\delta_{jq}$, that is we set a unit entry to the element corresponding to $x_i^{(j)}$ and zero everywhere else. The vector $\delta\boldsymbol{x}$ thus constructed can be used to probe the diagonal entry corresponding to the degree of freedom in question as:

$$d = \delta\boldsymbol{x}^T\mathbf{K}\delta\boldsymbol{x} = -\delta\boldsymbol{x}^T\delta\boldsymbol{f}$$

where $\delta\boldsymbol{f}$ is the force differential incurred by the constructed displacement. Additionally, if $\delta\mathbf{P}$ is the stress differential corresponding to this particular displacement $\delta\boldsymbol{x}$, section 5 of the paper establishes that

$$-\delta\boldsymbol{x}^T\delta\boldsymbol{f} = V_e(\delta\mathbf{F} : \mathcal{T} : \delta\mathbf{F}).$$

where $\mathcal{T} = \partial\mathbf{P}/\partial\mathbf{F}$. We will leverage this expression, in order to compute the diagonal element. For simplicity we will not concern ourselves with the global scalefactor V_e, which can simply be applied to the entire constructed system, and will compute the diagonal entry using

$$d = \delta\mathbf{F} : \mathcal{T} : \delta\mathbf{F} = \delta\mathbf{P} : \delta\mathbf{F} = (\mathbf{R}^T\delta\mathbf{P}) : (\mathbf{R}^T\delta\mathbf{F}) = \delta\hat{\mathbf{P}} : \delta\hat{\mathbf{F}} =$$

$$= \delta\hat{\mathbf{P}}_{\text{sym}} : \delta\hat{\mathbf{F}}_{\text{sym}} + \delta\hat{\mathbf{P}}_{\text{skew}} : \delta\hat{\mathbf{F}}_{\text{skew}}$$

where the last equation is due to the decoupled action of the tensor defined as $\hat{\mathcal{T}}$ in section 5, across the symmetric and skew symmetric subspaces.

Since $\delta x_p^{(q)} = \delta_{ip}\delta_{jq}$, we have:

$$\delta F_{qr} = \sum_p G_{rp}\delta x_p^{(q)} = \sum_p G_{rp}\delta_{ip}\delta_{jq} = G_{ri}\delta_{jq}$$

Thus

$$[\delta\hat{\mathbf{F}}]_{pr} = [\mathbf{R}^T\delta\mathbf{F}]_{pr} = \sum_q R_{qp}\delta F_{qr} = \sum_q R_{qp}G_{ri}\delta_{jq} = R_{jp}G_{ri}.$$

This implies that $\delta\hat{\mathbf{F}} = \boldsymbol{r}\boldsymbol{g}^T$, where $\boldsymbol{r}^T$ is the j-th row of $\mathbf{R}$, and $\boldsymbol{g}$ (which is a constant) is the i-th row of $\mathbf{G}$.

Finally we observe that

$$\mathcal{E}^T : \delta\hat{\mathbf{F}}_{\text{skew}} = \mathcal{E}^T : \delta\hat{\mathbf{F}} = \mathcal{E}^T : (\boldsymbol{r}\boldsymbol{g}^T) = (\mathcal{E} : \boldsymbol{g})\boldsymbol{r}$$

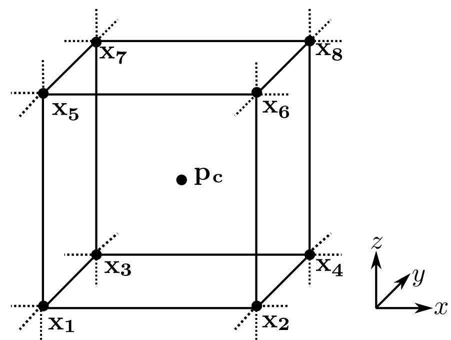

Figure 1: *Geometric configuration of an individual cube element* Ω_e

where the last equality is due to the symmetries of the alternating tensor. In our paper we used the notation $\mathbf{M}_i = \mathcal{E} : \boldsymbol{g}$, thus the last equation becomes

$$\mathcal{E}^T : \delta\hat{\mathbf{F}}_{\text{skew}} = \mathbf{M}_i \boldsymbol{r}.$$

We can now apply the formulations of section 5, to obtain

$$\delta\hat{\mathbf{P}}_{\text{sym}} : \delta\hat{\mathbf{F}}_{\text{sym}} = \lambda \text{tr}(\delta\hat{\mathbf{F}}_{\text{sym}})\mathbf{I} : \delta\hat{\mathbf{F}}_{\text{sym}} = \lambda \text{tr}^2(\delta\hat{\mathbf{F}}_{\text{sym}}) =$$
$$= \lambda \text{tr}^2(\delta\hat{\mathbf{F}}) = \lambda \text{tr}^2(\boldsymbol{r}\boldsymbol{g}^T) = \boldsymbol{r}^T(\lambda \boldsymbol{g}\boldsymbol{g}^T)\boldsymbol{r}$$

$$\delta\hat{\mathbf{P}}_{\text{skew}} : \delta\hat{\mathbf{F}}_{\text{skew}} = \left(\mathcal{E}^T : \delta\hat{\mathbf{F}}_{\text{skew}}\right)^T \mathbf{L} \left(\mathcal{E}^T : \delta\hat{\mathbf{F}}_{\text{skew}}\right) = \boldsymbol{r}^T \mathbf{M}_i^T \mathbf{L} \mathbf{M}_i \boldsymbol{r}$$

Adding these two expressions, the diagonal term becomes

$$d = \boldsymbol{r}^T \mathbf{N}_i \boldsymbol{r}, \text{ where } \mathbf{N}_i = \lambda \boldsymbol{g}\boldsymbol{g}^T + \mathbf{M}_i^T \mathbf{L} \mathbf{M}_i$$

As we noted in the paper, $\mathbf{N}_i$ does not vary across elements, and can be precomputed and stored. Finally, we have not added the contribution of the Laplace term Ψ_Δ to the diagonal part; as we discussed, intuitively this term generates the standard 7-point discretization of the scaled Laplace operator $-2\mu\Delta$ on every component, and every node of the grid. The central point of this scaled 7-point Laplacian equals $12\mu/h^2$, and is equally distributed to the 8 incident elements of each node, for a contribution of $(3\mu)/(2h^2)$ on each one. These values can, of course, be verified directly, by computing the Hessian of the discrete term Ψ_Δ as defined in section 4. Ultimately, with the addition of the Laplace term, the element contribution to the degree of freedom $x_i^{(j)}$ becomes:

$$d_{\text{total}} = \frac{3\mu}{2h^2} + \boldsymbol{r}^T \mathbf{N}_i \boldsymbol{r}$$